

Kantorovich's theorem on Newton's method for solving strongly regular generalized equation

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Abstract

In this paper we consider the Newton's method for solving the generalized equation of the form $f(x) + F(x) \ni 0$, where $f : \Omega \rightarrow Y$ is a continuously differentiable mapping, X and Y are Banach spaces, $\Omega \subseteq X$ an open set and $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty closed graph. We show that, under strong regularity of the generalized equation, concept introduced by S. M. Robinson in [27], and starting point satisfying the Kantorovich's assumptions, the Newton's method is quadratically convergent to a solution, which is unique in a suitable neighborhood of the starting point. The analysis presented based on Banach Perturbation Lemma for generalized equation and the majorant technique, allow to unify some results pertaining the Newton's method theory.

Keywords: Generalized equation. Newton's method. majorant condition. semi-local convergence.

1 Introduction

In this paper we consider the Newton's method for solving the generalized equation of the form

$$f(x) + F(x) \ni 0, \quad (1)$$

where $f : \Omega \rightarrow \mathbb{Y}$ is a continuously differentiable mapping, \mathbb{X} and \mathbb{Y} are Banach spaces, $\Omega \subseteq \mathbb{X}$ is an open set and $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping with closed nonempty graph. As is well known, the generalized equation (1) is an abstract model for various problems in classical analysis and its applications. For instance, if $\mathbb{X} = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^{p+q}$ and $F = \mathbb{R}^p_- \times \{0\}$ is the product of the nonpositive orthant in \mathbb{R}^p with the origin in \mathbb{R}^q , then the inclusion (1) describes a system of equalities and inequalities. If F is the normal cone mapping N_C of a convex set C in \mathbb{Y} , then the inclusion (1) is the variational inequality problem, which covers wide range of problems in mathematical programming. Additional comments about generalized equations can be found in [1, 2, 8, 10, 11, 12, 13, 18, 25] and the references cited therein.

Newton's method to solve (1) formally generates a sequence, for an initial point x_0 , as follows

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \ni 0, \quad k = 0, 1, \dots \quad (2)$$

This method may be viewed as a Newton-type method based on a partial linearization, which has been studied in several papers including [1, 2, 8, 12]; see also [10, Section 6C]. When $F \equiv 0$, the iteration (2) becomes the standard Newton's method for solving the nonlinear equation $f(x) = 0$. If $\mathbb{X} = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^m$ and $F = \mathbb{R}^s_- \times \{0\}^{m-s}$, then (2) is a Newton's method for solving a system of equalities and inequalities; see [6]. Now, if F is the normal cone mapping N_C , of a convex set C in \mathbb{Y} and $\mathbb{Y} = \mathbb{X}^*$, then (2) is the

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known version of the Newton's method for solving variational inequality; see [7, 18]. In particular, if (1) represents the Karush-Kuhn-Tucker optimality conditions for a mathematical programming problem, then the procedure (2) describes the well-known sequential quadratic programming method; see for example [10, pag. 334].

L. V. Kantorovich in [19], see also [20, 23], was the first to prove a convergence result for Newton's method for solving the equation $f(x) = 0$, where $f : \Omega \rightarrow \mathbb{Y}$ is a continuously differentiable mapping, \mathbb{X} and \mathbb{Y} are Banach spaces and $\Omega \subseteq \mathbb{X}$ is an open set. Using conditions on x_0 the starting point, namely, under the condition that $f'(x_0)^{-1}$ exists and $\|f'(x_0)^{-1}f(x_0)\|$ is bounded, L.V. Kantorovich obtained well definition of the method, quadratic convergence and uniqueness of solution. The idea employed in the proof of convergence was the technique of majorization, which consists in bound the Newton's sequence by a scalars sequence. This technique has been used and extended for various researchers, including [5, 13, 15, 16, 17, 24, 30, 32]. S. M. Robinson in [25], using the idea of convex process introduced by Rockafellar [29], see also [26, 28], established a generalization of the Kantorovich's theorem for solving the inclusion $f(x) \in C$, where $f : \Omega \rightarrow \mathbb{Y}$ is a continuously differentiable mapping, \mathbb{X} and \mathbb{Y} are Banach spaces, $\Omega \subseteq \mathbb{X}$ is an open set and $C \subseteq \mathbb{Y}$ is a nonempty closed and convex cone. The paper [25] has been extended for various authors, see for instance [5, 13, 15, 21]. In his Ph.D. thesis, N. H. Josephy in [18] studied Newton's method for solving the variational inequality $f(x) + N_C \ni 0$, where $f : \Omega \rightarrow \mathbb{R}^m$ is a continuously differentiable mapping, $\Omega \subseteq \mathbb{R}^n$ is an open set and N_C is the normal cone mapping of a convex set $C \subset \mathbb{R}^m$. For guarantee the well definition of the method, *strong regularity* property on $f(x) + N_C$, concept introduced in the theory of generalized equations by S.M. Robinson in [27], was used. If $\mathbb{X} = \mathbb{Y}$ and $N_C = \{0\}$, then strong regularity is equivalent to $f'(x)^{-1}$ be a continuous linear operator. If $\mathbb{X} = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^m$ and $F = \mathbb{R}_+^s \times \{0\}^{m-s}$, then strong regularity is equivalent to Mangasarian-Fromovitz constraint qualification; see [10, Example 4D.3]. An important case is when (1) represents the Karush-Kuhn-Tucker's systems for the standard nonlinear programming problem with a strict local minimizer, see [10] pag. 232. In this case, the strong regularity of this system is equivalent to the linear independence of the gradients of the active constraints and the strong second-order sufficient optimality condition; see [9, Theorem 6].

The usual hypotheses used to obtain quadratic convergence of Newton's method (2), for solving equation (1), is the Lipschitz continuity of f' in a neighborhood of an initial point; see [5, 7, 8, 13, 15, 16, 18]. Indeed, keeping control of the derivative is an important point in the convergence analysis of Newton's method. On the other hand, a couple of papers have dealt with the issue of convergence analysis of the Newton's method by relaxing the assumption of Lipschitz continuity of f' , see for example [16, 30, 31], actually all this conditions are equivalent to X. Wang's condition introduced in [30]. The advantage of working with a *majorant condition* relaxing the assumption of Lipschitz continuity of f' rests in the fact that it allow to unify several convergece results pertaining to Newton's method; see [16, 30]. In this paper we rephrase the majorant condition introduced in [16], in order to study the convergence properties of Newton's method (2). The analysis presented provides a clear relationship between the majorant function and the function defining generalized equation (1). Also, it allows us to obtain the convergence radius for the method, bound for its convergence rates with respect to the majorant condition and uniqueness of solution. The convergence analysis of the Newton's method under Lipschitz's and Smale's conditions, are provided as special case. Up to our knowledge, this is the first time that the Newton's method to solving generalized equations under Smale's condition in the starting point is analyzed. In addition, it is worth mentioning that the recent approach for analyzing semi-local convergence of Newton's method and its variants, for solving generalized equation, use contraction mapping principle for set-valued mappings, see [5, 7, 8], while our approach is based in the Banach Perturbation Lemma. In this sense, our approach is related to the techniques used in [4, 7, 18].

The organization of the paper is as follows. In Section 2, some notations and important results used throughout the paper are presented. In Section 3, the main result is stated and in Section 3.1 properties of the majorant function, the main relationships between the majorant function and the nonlinear operator are established. In Section 3.2 the main result is proved and the uniqueness of the solution and some

applications of this result are given in Section 4. Some final remarks are made in Section 5.

2 Preliminaries

The following notations and results are used throughout our presentation. We begin with the following elementary convex analysis result:

Proposition 1. *Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \rightarrow \mathbb{R}$ be a convex function. If $s, t, r \in I$, $s < r$, and $s \leq t \leq r$ then $\varphi(t) - \varphi(s) \leq [\varphi(r) - \varphi(s)] [(t - s)/(r - s)]$. Moreover, if φ is continuously differentiable then $\varphi' : I \rightarrow \mathbb{R}$ is increasing and, for any $s_0 \in \text{int}(I)$, there holds*

$$\varphi'(s_0) := \lim_{s \rightarrow s_0^-} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s} = \sup_{s < s_0} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s}.$$

Let \mathbb{X}, \mathbb{Y} be Banach spaces, the *open* and *closed balls* at x with radius $\delta \geq 0$ are denoted, respectively, by $B(x, \delta) = \{y \in X : \|x - y\| < \delta\}$ and $B[x, \delta] = \{y \in X : \|x - y\| \leq \delta\}$. We denote by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ the *space consisting of all continuous linear mappings* $A : \mathbb{X} \rightarrow \mathbb{Y}$ and the *norm* of A is defined by $\|A\| := \sup \{\|Ax\| : \|x\| \leq 1\}$. Let $\Omega \subseteq \mathbb{X}$ and $h : \Omega \rightarrow \mathbb{Y}$ a function with Fréchet derivative at all $x \in \text{int}(\Omega)$. The Fréchet derivative of h at x is the linear mapping $h'(x) : \mathbb{X} \rightarrow \mathbb{Y}$ which is continuous. We identify as the *graph* of the set-valued mapping $H : \mathbb{X} \rightrightarrows \mathbb{Y}$ the set $\text{gph } H := \{(x, y) \in \mathbb{X} \times \mathbb{Y} : y \in H(x)\}$. The *domain* and the *range* of H are, respectively, the sets $\text{dom } H = \{x \in \mathbb{X} : H(x) \neq \emptyset\}$ and $\text{rge } H = \{y \in \mathbb{Y} : y \in H(x) \text{ for some } x\}$. The *inverse* of H is the set-valued mapping $H^{-1} : \mathbb{Y} \rightrightarrows \mathbb{X}$ defined by $H^{-1}(y) = \{x \in \mathbb{X} : y \in H(x)\}$.

Definition 1. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, Ω be an open nonempty subset of \mathbb{X} , $h : \Omega \rightarrow \mathbb{Y}$ be a Fréchet differentiable with derivative h' and $H : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping. The *partial linearization* of the mapping $h + H$ at $x \in \mathbb{X}$ is the set-valued mapping $L_h(x, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ given by*

$$L_h(x, y) := h(x) + h'(x)(y - x) + H(y). \quad (3)$$

For each $x \in \mathbb{X}$, the inverse $L_h(x, \cdot)^{-1} : \mathbb{Y} \rightrightarrows \mathbb{X}$ of the mapping $L_h(x, \cdot)$ at $z \in Y$ is denoted by

$$L_h(x, z)^{-1} := \{y \in X : z \in h(x) + h'(x)(y - x) + H(y)\}. \quad (4)$$

Remark 1. *If in the above definition we have $H \equiv 0$, $z = 0$ and $h'(x)$ invertible, then the inverse mapping $x \mapsto L_h(x, 0)^{-1} = x - h'(x)^{-1}h(x)$ is the well known Newton's iteration mapping for solving the equation $h(x) = 0$.*

An important element in the analysis of Newton's method for solving the equation $f(x) = 0$, is the behavior of inverse $f'(x)^{-1}$ for x in a neighborhood of an initial point. The analogous element for the generalized equation (1) is the behavior of the set-valued mapping $L_f(x, 0)^{-1}$, for x in a neighborhood of an initial point. It is worth point out that, N. H. Josephy in [18] was the first to consider Newton's method for solving the generalize equation $f(x) + N_C(x) \ni 0$, where C is the normal cone of a convex set $C \subset \mathbb{R}^n$, by defining the Newton's iteration as $L_f(x_k, 0)^{-1} \ni x_{k+1}$ for $k = 0, 1, \dots$, which is equivalent to (2), to the particular case $F = N_C$. N. H. Josephy in [18], for analyzing Newton's method, employed the important concept of strong regularity defined by S.M. Robinson [27], which assuring “good behavior” of $L_f(x, 0)^{-1}$ for x in a suitable neighborhood of an initial point x_0 . Here we adopt the following definition due to Robinson given in [27].

Definition 2. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, Ω be an open nonempty subset of \mathbb{X} , $h : \Omega \rightarrow \mathbb{Y}$ be Fréchet differentiable with derivative h' and $H : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping. The mapping $h + H$ is said to be *strongly regular* at x for y , when $y \in h(x) + H(x)$ and there exist constants $r_x > 0$, $r_y > 0$ and $\lambda > 0$*

such that $B(x, r_x) \subset \Omega$, the mapping $z \mapsto L_h(x, z)^{-1} \cap B(x, r_x)$ is a single-valued from the ball $B(y, r_y)$ to $B(x, r_x)$, which is Lipschitzian on $B(y, r_y)$ with modulus λ , i.e.,

$$\|L_h(x, u)^{-1} \cap B(x, r_x) - L_h(x, v)^{-1} \cap B(x, r_x)\| \leq \lambda \|u - v\|, \quad \forall u, v \in B(y, r_y).$$

In this case, we refer to λ as the Lipschitz constant.

Since the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$, for simplify the notation we are using in above definition $w = L_f(x, 0)^{-1} \cap B(x_1, r_{x_1})$ instead of $\{w\} := L_f(x, 0)^{-1} \cap B(x_1, r_{x_1})$. From now on we will use this simplified notation.

Remark 2. If $H(x) \equiv \{0\}$ then the property of $h + H \equiv h$ be strongly regular at x for y , reduces to $h'(x)$ has an inverse $h'(x)^{-1}$. Moreover, in this case, the strongly regular radii associated to $h + H$ at x for y are given by $r_x = +\infty$ and $r_y = +\infty$, respectively, and the Lipschitz constant is $\lambda = \|h'(x)^{-1}\|$.

For a detailed discussion about the Definition 2; see [10, 27]. The next result is a type of implicit function theorem for generalized equations satisfying the strongly regular condition and its proof is an immediate consequence of [10, Theorem 5F.4] on page 294; see also [27, Theorem 2.1].

Theorem 2. Let \mathbb{X}, \mathbb{Y} and \mathbb{Z} be Banach spaces, $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping and $g : \mathbb{Z} \times \mathbb{X} \rightarrow \mathbb{Y}$ be a continuous function, having partial Fréchet derivative with respect the second variable $D_x g$ on $\mathbb{Z} \times \mathbb{X}$, which is also continuous. Let $\bar{p} \in \mathbb{Z}$ and suppose that \bar{x} solves the generalized equation

$$g(\bar{p}, x) + G(x) \ni 0. \quad (5)$$

Assume that the mapping $g(\bar{p}, \cdot) + G$ is strongly regular at \bar{x} for 0, with associated Lipschitz constant λ . Then, for any $\epsilon > 0$ there exist $r_{\bar{p}} > 0$ and $r_{\bar{x}} > 0$, which depend of ϵ , and a single-valued mapping $s : B(\bar{p}, r_{\bar{p}}) \rightarrow B(\bar{x}, r_{\bar{x}})$ such that for any $p \in B(\bar{p}, r_{\bar{p}})$, $s(p)$ is the unique solution in $B(\bar{x}, r_{\bar{p}})$ of the inclusion $g(p, x) + G(x) \ni 0$, and $s(\bar{p}) = \bar{x}$. Moreover, there holds

$$\|s(p') - s(p)\| \leq (\lambda + \epsilon) \|g(p', s(p)) - g(p, s(p))\|, \quad \forall p, p' \in B(\bar{p}, r_{\bar{p}}).$$

Indeed, the first version of the Theorem 2 was proved by S.M.Robinson; see [27, Theorem 2.1], to the particular case $F = N_C$, where C is the normal cone of a convex set $C \subset \mathbb{X}$ and, as an application, a version involving the normal cone of the Banach Perturbation Lemma for linear operator was obtained; see [27, Theorem 2.4]. N. H. Josephy in [18], used this Banach Pertubation Lemma; see [18, Corollary 1], for proving that the Newton iteration

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni 0, \quad k = 0, 1, \dots$$

where C is the normal cone of a convex set $C \subset \mathbb{R}^n$, is well defined and quadratically convergent for a solution of the inclusion $f(x) + N_C(x) \ni 0$. In the next lemma we apply Theorem 2 to obtain a version, involving a general set-valued mapping, of the Banach Perturbation Lemma for linear operator. The proof of this result is similar to the correspondent one [18, Corollary 1].

Lemma 3. Let \mathbb{X}, \mathbb{Y} be Banach spaces, a_0 be a point of \mathbb{Y} , $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping and $A_0 : \mathbb{X} \rightarrow \mathbb{Y}$ be a bounded linear mapping. Suppose that \bar{x} is a point of \mathbb{X} which satisfies the generalized equation

$$0 \in A_0 x + a_0 + G(x).$$

Assume that the mapping $A_0 + a_0 + G$ is strongly regular at \bar{x} for 0 with Lipschitz constant λ . Then, there exist $r_{\bar{x}} > 0$, $r_{A_0} > 0$, $r_{a_0} > 0$ and $r_0 > 0$ such that, for any $A \in B(A_0, r_{A_0}) \subset \mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $a \in B(a_0, r_{a_0}) \subset \mathbb{Y}$ letting $T(A, a, \cdot) : B(\bar{x}, r_{\bar{x}}) \rightrightarrows \mathbb{Y}$ be defined as

$$T(A, a, x) := Ax + a + G(x),$$

the mapping $y \mapsto T(A, a, y)^{-1} \cap B(\bar{x}, r_{\bar{x}})$ is a single-valued mapping from $B(0, r_0) \subset \mathbb{Y}$ to $B(\bar{x}, r_{\bar{x}})$. Moreover, for each $A \in B(A_0, r_{A_0})$ and $a \in B(a_0, r_{a_0})$ there holds $\lambda \|A - A_0\| < 1$ and the mapping $y \mapsto T(A, a, y)^{-1} \cap B(\bar{x}, r_{\bar{x}})$ is also Lipschitzian on $B(0, r_0)$ as follows

$$\|T(A, a, y_1)^{-1} \cap B(\bar{x}, r_{\bar{x}}) - T(A, a, y_2)^{-1} \cap B(\bar{x}, r_{\bar{x}})\| \leq \frac{\lambda}{1 - \lambda \|A - A_0\|} \|y_1 - y_2\|, \quad \forall y_1, y_2 \in B(0, r_0).$$

Proof. Let $\mathbb{Z} = \mathcal{L}(\mathbb{X}, \mathbb{Y}) \times \mathbb{Y}$ and $g : \mathbb{Z} \times \mathbb{X} \rightarrow \mathbb{Y}$ be an operator defined by $g(A, a, x) = Ax + a$. The operator g is continuous on $\mathbb{Z} \times \mathbb{X}$ and has partial Fréchet derivative with respect to the variable x given by $D_x g(A, a, x) = A$. Note that

$$A_0 \bar{x} + a_0 + G(\bar{x}) = g(A_0, a_0, x) + D_x g(A_0, a_0, x)(\bar{x} - x) + G(\bar{x}), \quad \forall x \in \mathbb{X},$$

and, by assumption, the mapping $A_0 + a_0 + G$ is strongly regular at \bar{x} for 0 with Lipschitz constant λ . Then, we may apply Theorem 2 with $\mathbb{Z} = \mathcal{L}(\mathbb{X}, \mathbb{Y}) \times \mathbb{Y}$, $\bar{p} = (A_0, a_0)$, $p = (A, a)$ and $g(p, x) = Ax + a$, for concluding that, for any $\epsilon > 0$, there exist $r_{\bar{p}} > 0$ and $r_{\bar{x}} > 0$, which depend of ϵ , and a single-valued mapping $s : B(\bar{p}, r_{\bar{p}}) \rightarrow B(\bar{x}, r_{\bar{x}})$ such that for any $(A, a) \in B(\bar{p}, r_{\bar{p}})$, $s(A, a)$ is the unique solution in $B(\bar{x}, r_{\bar{x}})$ of the inclusion

$$T(A, a, x) := Ax + a + G(x) \ni 0,$$

and $s(A_0, a_0) = \bar{x}$. Moreover, the following inequality holds

$$\|s(A, a) - \bar{x}\| \leq (\lambda + \epsilon) \|(A - A_0)\bar{x} + (a - a_0)\|, \quad \forall (A, a) \in B(\bar{p}, r_{\bar{p}}).$$

Thus, the single-valued mapping s is bounded and we can choose $r_{A_0} > 0$, $r_{a_0} > 0$ and $r_0 > 0$ such that $B(A_0, r_{A_0}) \times [B(a_0, r_{a_0}) - B(0, r_0)] \subset B(\bar{p}, r_{\bar{p}})$, and for each $A \in B(A_0, r_{A_0})$, $a \in B(a_0, r_{a_0})$ and $y \in B(0, r_0)$ there holds

$$\lambda \|A - A_0\| < 1, \quad y + (A_0 - A)s(A, a) + (a_0 - a) \in B(0, \hat{r}_0),$$

where the radius $\hat{r}_0 > 0$ is given in the definition of strong regularity of $A_0 + a_0 + G$ at \bar{x} for 0. Let $A \in B(A_0, r_{A_0})$, $a \in B(a_0, r_{a_0})$ and $y_1, y_2 \in B(0, r_0)$, and let $s(A, a - y_1)$ and $s(A, a - y_2)$ be the solutions associated with y_1 and y_2 , respectively. Since $T(A, a, s(A, a - y_i)) \ni y_i$, i.e., $s(A, a - y_i) = T(A, a, y_i)^{-1} \cap B(\bar{x}, r_{\bar{x}})$, for $i = 1, 2$, after some manipulation, we obtain that

$$y_i + (A_0 - A)s(A, a - y_i) + (a_0 - a) \in A_0 s(A, a - y_i) + a_0 + G(s(A, a - y_i)), \quad i = 1, 2. \quad (6)$$

Therefore, taking into account that $A_0 + a_0 + G$ is strongly regular at \bar{x} for 0 with associated Lipschitz constant λ , the inclusions in (6) imply that

$$\begin{aligned} \|s(A, a - y_1) - s(A, a - y_2)\| &\leq \\ &\lambda \|[y_1 + (A_0 - A)s(A, a - y_1) + (a_0 - a)] - [y_2 + (A_0 - A)s(A, a - y_2) + (a_0 - a)]\|. \end{aligned}$$

Using properties of the norm, last inequality becomes to

$$\|s(A, a - y_1) - s(A, a - y_2)\| \leq \lambda \|y_1 - y_2\| + \lambda \|A_0 - A\| \|s(A, a - y_1) - s(A, a - y_2)\|.$$

Now, since $\lambda \|A - A_0\| < 1$ for each $A \in B(A_0, r_{A_0})$, then last inequality implies that

$$\|s(A, a - y_1) - s(A, a - y_2)\| \leq \frac{\lambda}{1 - \lambda \|A - A_0\|} \|y_1 - y_2\|,$$

and the result follows by noting that $s(A, a - y) = T(A, a, y)^{-1} \cap U$ and $y_1, y_2 \in B(0, r_0)$ are arbitrary. \square

Next we establish a corollary to Lemma 3, which will have important rule in the sequel.

Corollary 4. Let \mathbb{X}, \mathbb{Y} be Banach spaces, Ω be an open nonempty subset of \mathbb{X} , $f : \Omega \rightarrow \mathbb{Y}$ be continuous with Fréchet derivative f' continuous, and $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping. Suppose that $x_0 \in \Omega$ and the mapping $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ is strongly regular at x_1 for 0 with associated Lipschitz constant $\lambda > 0$. Then, there exist three constants $r_{x_1}, r_0 > 0$ and $r_{x_0} > 0$ such that, for each $x \in B(x_0, r_{x_0})$, there holds $\lambda \|f'(x) - f'(x_0)\| < 1$, the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$ and Lipschitzian as follows

$$\|L_f(x, u)^{-1} \cap B(x_1, r_{x_1}) - L_f(x, v)^{-1} \cap B(x_1, r_{x_1})\| \leq \frac{\lambda}{1 - \lambda \|f'(x) - f'(x_0)\|} \|u - v\|, \quad \forall u, v \in B(0, r_0).$$

Proof. Since $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ is strongly regular at x_1 for 0 with associated Lipschitz constant $\lambda > 0$, applying first part of Lemma 3 with $\bar{x} = x_1$, $A_0 = f'(x_0)$, $a_0 = f(x_0) - f'(x_0)x_0$ and $G = F$, we conclude that there exist $r_{x_1} > 0$, $\tilde{r} > 0$, $\hat{r} > 0$ and $r_0 > 0$ such that, for any $A \in B(f'(x_0), \tilde{r}) \subset \mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $a \in B(f(x_0) - f'(x_0)x_0, \hat{r}) \subset \mathbb{Y}$, letting $T(A, a, \cdot) : B(x_1, r_{x_1}) \rightrightarrows \mathbb{Y}$ be defined as

$$T(A, a, y) := Ay + a + F(y),$$

the mapping $z \mapsto T(A, a, z)^{-1} \cap B(x_1, r_{x_1})$ is a single-valued mapping from $B(0, r_0)$ to $B(x_1, r_{x_1})$. Due to f be continuous with f' continuous, then there exists $r_{x_0} > 0$ such that $\lambda \|f'(x) - f'(x_0)\| < 1$,

$$f'(x) \in B(f'(x_0), \tilde{r}), \quad f(x) - f'(x)x \in B(f(x_0) - f'(x_0)x_0, \hat{r}), \quad \forall x \in B(x_0, r_{x_0}).$$

Hence, we conclude that for each $x \in B(x_0, r_{x_0})$, the mapping $z \mapsto T(f'(x), f(x) - f'(x)x, z)^{-1} \cap B(x_1, r_{x_1})$ is a single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$, where

$$T(f'(x), f(x) - f'(x)x, y) := f'(x)y + f(x) - f'(x)x + F(y) = f(x) + f'(x)(y - x) + F(y). \quad (7)$$

Since Definition 1 and (7) imply that $L_f(x, y) = T(f'(x), f(x) - f'(x)x, y)$, for all $x \in B(x_0, r_{x_0})$ and $y \in B(x_1, r_{x_1})$, after some manipulations we have, for each $z \in B(0, r_0)$,

$$L_f(x, z)^{-1} \cap B(x_1, r_{x_1}) = T(f'(x), f(x) - f'(x)x, z)^{-1} \cap B(x_1, r_{x_1}), \quad \forall x \in B(x_0, r_{x_0}). \quad (8)$$

Therefore, for each $x \in B(x_0, r_{x_0})$, the last equality and (7) imply that $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$, which proof the first part of corollary. Finally, taking into account (8) and second part of Lemma 3, we also conclude that the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is Lipschitzian from $B(0, r_0)$ to $B(x_1, r_{x_1})$ with Lipschitz constant $\lambda/[1 - \lambda \|f'(x) - f'(x_0)\|]$, which conclude the proof. \square

Remark 3. If in above corollary we have $F \equiv 0$. Then, for each $x \in B(x_0, r_{x_0})$, the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ be single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$ means that $f'(x)$ is invertible and, for each $z \in B(0, r_0)$, there exists a unique $y \in B(x_1, r_{x_1})$ such that $y = L_f(x, z)^{-1} \cap B(x_1, r_{x_1}) = x + f'(x)^{-1}(z - f(x))$. Moreover, $\lambda \|f'(x) - f'(x_0)\| < 1$ and there holds

$$\|f'(x)^{-1}(u - v)\| = \|[x + f'(x)^{-1}(u - f(x))] - [x + f'(x)^{-1}(v - f(x))]\| \leq \frac{\lambda \|u - v\|}{1 - \lambda \|f'(x) - f'(x_0)\|},$$

for all $u, v \in \mathbb{Y}$ and $x \in B(x_0, r_{x_0})$. Therefore, from Remark 2, we have $r_{x_1} = r_0 = +\infty$ and $\lambda = \|f'(x_0)^{-1}\|$ and, last inequality becomes

$$\|f'(x)^{-1}\| \leq \frac{\|f'(x_0)^{-1}\|}{1 - \|f'(x_0)^{-1}\| \|f'(x) - f'(x_0)\|}, \quad \forall x \in B(x_0, r_{x_0}).$$

3 Kantorovich's theorem for Newton's method

In this section, our goal is to state and prove a Kantorovich's theorem for Newton's method for solving the generalized equation of the form (1). To state the theorem we need to fix some important constants. Let \mathbb{X} , \mathbb{Y} be Banach spaces, Ω be an open nonempty subset of \mathbb{X} , $f : \Omega \rightarrow \mathbb{Y}$ be continuous with Fréchet derivative f' continuous, and $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping. From now on, for $x_0 \in \Omega$ and a partial linearization mapping $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at x_0 , given by

$$L_f(x_0, x) := f(x_0) + f'(x_0)(x - x_0) + F(x),$$

strongly regular at x_1 for 0 with associated Lipschitz constant λ , we refer to the real numbers

$$r_{x_1} > 0, \quad r_0 > 0, \quad r_{x_0} > 0, \quad (9)$$

as the three constants given by Corollary 4. The statement of main result is:

Theorem 5. *Let \mathbb{X} , \mathbb{Y} be Banach spaces, $\Omega \subseteq \mathbb{X}$ an open set, $f : \Omega \rightarrow \mathbb{Y}$ be continuous with Fréchet derivative f' continuous and $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping with closed graph. Let $x_0 \in \Omega$, $R > 0$ and $\kappa := \sup\{t \in [0, R) : B(x_0, t) \subset \Omega\}$. Suppose that the partial linearization mapping $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at x_0 , is strongly regular at $x_1 \in \Omega$ for 0 with associated Lipschitz constant $\lambda > 0$ and there exist $\psi : [0, R) \rightarrow \mathbb{R}$ twice continuously differentiable function such that*

$$\lambda \|f'(y) - f'(x)\| \leq \psi'(\|y - x\| + \|x - x_0\|) - \psi'(\|x - x_0\|), \quad (10)$$

for all $x, y \in B(x_0, \kappa)$ and $\|y - x\| + \|x - x_0\| < R$. Moreover, suppose that

$$\|x_1 - x_0\| \leq \psi(0), \quad (11)$$

and the following conditions hold:

h1) $\psi(0) > 0$, $\psi'(0) = -1$;

h2) ψ' is convex and strictly increasing;

h3) $\psi(t) = 0$ for some $t \in (0, R)$ and let $t_* := \min\{t \in [0, R) : \psi(t) = 0\}$.

Additionally, for the constants r_0 and r_{x_0} fixed in (9), suppose that the following inequalities hold:

$$t_* \leq r_{x_0}, \quad \frac{\psi''(t_*)}{2\lambda} \psi(0)^2 < r_0. \quad (12)$$

Then, the sequences generated by Newton's method for solving the generalized equation $0 \in f(x) + F(x)$ and the equation $\psi(t) = 0$, with starting point x_0 and $t_0 = 0$, defined respectively by,

$$x_{k+1} := L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}), \quad t_{k+1} = t_k - \psi(t_k)/\psi'(t_k), \quad k = 0, 1, \dots, \quad (13)$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $(0, t_*)$ and converges to t_* , $\{x_k\}$ is contained in $B(x_0, t_*)$ and converges to the point $x_* \in B[x_0, t_*]$, which is the unique solution of the generalized equation $0 \in f(x) + F(x)$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$. Moreover, $\{x_k\}$ and $\{t_k\}$ satisfies

$$\|x_* - x_k\| \leq t_* - t_k, \quad \|x_* - x_{k+1}\| \leq \frac{t_* - t_{k+1}}{(t_* - t_k)^2} \|x_* - x_k\|^2, \quad (14)$$

for all $k=0, 1, \dots$, and the sequences $\{x_k\}$ and $\{t_k\}$ converge Q -linearly as follows

$$\|x_* - x_{k+1}\| \leq \frac{1}{2} \|x_* - x_k\|, \quad t_* - t_{k+1} \leq \frac{1}{2} (t_* - t_k), \quad k = 0, 1, \dots \quad (15)$$

Additionally, if the following condition holds

h4) $\psi'(t_*) < 0$,

then the sequences, $\{x_k\}$ and $\{t_k\}$ converge Q -quadratically as follows

$$\|x_* - x_{k+1}\| \leq \frac{\psi''(t_*)}{-2\psi'(t_*)} \|x_* - x_k\|^2, \quad t_* - t_{k+1} \leq \frac{\psi''(t_*)}{-2\psi'(t_*)} (t_* - t_k)^2, \quad k = 0, 1, \dots \quad (16)$$

Remark 4. In Section 4, we will present several particular instances of Theorem 5, by presenting the explicit majorant function. For instance, when $F \equiv \{0\}$ and f' satisfies a Lipschitz-type condition, i.e., the majorant function associated to f' is a quadratic polynomial defined by the Lipschitz constant, we retrieve a version of the classical Kantorovich's theorem on Newton's method; for example, see [19, 20].

Henceforward we assume that all the assumptions in Theorem 5 holds.

3.1 Basic results

In this section we will establish some results about the majorant function $\psi : [0, R) \rightarrow \mathbb{R}$ and, some relationships between the majorant function and the set-valued mapping $f + F$. We begin by reminding that Proposition 3 of [16] state that the majorant function ψ has a smallest root $t_* \in (0, R)$, is strictly convex, $\psi(t) > 0$ and $\psi'(t) < 0$, for all $t \in [0, t_*)$. Moreover, $\psi'(t_*) \leq 0$ and $\psi'(t_*) < 0$ if, and only if, there exists $t \in (t_*, R)$ such that $\psi(t) < 0$. Since $\psi'(t) < 0$ for all $t \in [0, t_*)$, the Newton iteration of the majorant function ψ is well defined in $[0, t_*)$. Let us call it $n_\psi : [0, t_*) \rightarrow \mathbb{R}$ such that

$$n_\psi(t) = t - \frac{\psi(t)}{\psi'(t)}. \quad (17)$$

The next result will be used to obtain the convergence rate of the sequence generated by Newton's method for solving $\psi(t) = 0$. Its proof can be found in [16, Proposition 4].

Lemma 6. For all $t \in [0, t_*)$ we have $n_\psi(t) \in [0, t_*)$, $t < n_\psi(t)$ and $t_* - n_\psi(t) \leq \frac{1}{2}(t_* - t)$. Moreover, the Newton step function $[0, t_*) \mapsto -\psi(t)/\psi'(t) \in [0, +\infty)$ is decreasing. If ψ also satisfies **h4** then

$$t_* - n_\psi(t) \leq \frac{D^-\psi'(t_*)}{-2\psi'(t_*)} (t_* - t)^2, \quad \forall t \in [0, t_*).$$

Using (17), the definition of $\{t_k\}$ in (13) is equivalent to the following one

$$t_0 = 0, \quad t_{k+1} = n_\psi(t_k), \quad k = 0, 1, \dots \quad (18)$$

The next result contain the main convergence properties of the above sequence and its prove, which is a consequence of Lemma 6, follows the same pattern as the proof of Corollary 2.15 of [14].

Corollary 7. The sequence $\{t_k\}$ is well defined, strictly increasing and is contained in $[0, t_*)$. Moreover, $\{t_k\}$ converges Q -linearly to t_* as the second inequality in (15). Additionally, If **h4** holds, then $\{t_k\}$ converges Q -quadratically to t_* as the second inequality in (16) and converges Q -quadratically.

Therefore, we have obtained all the statements about the majorant sequence $\{t_k\}$ in Theorem 5. Now, we are going to establish some relationships between the majorant function and the set-valued mapping $f + F$. The next result is a consequence of Corollary 4.

Proposition 8. For any $x \in B(x_0, t_*)$, the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$ and there holds

$$\|L_f(x, u)^{-1} \cap B(x_1, r_{x_1}) - L_f(x, v)^{-1} \cap B(x_1, r_{x_1})\| \leq -\frac{\lambda}{\psi'(\|x - x_0\|)} \|u - v\|, \quad \forall u, v \in B(0, r_0).$$

Proof. Definitions of the constants r_{x_1} , r_0 and r_{x_0} in (9) together with Corollary 4 imply that, for any $x \in B(x_0, r_{x_0})$, the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$ and Lipschitzian as follows

$$\|L_f(x, u)^{-1} \cap B(x_1, r_{x_1}) - L_f(x, v)^{-1} \cap B(x_1, r_{x_1})\| \leq \frac{\lambda}{1 - \lambda\|f'(x) - f'(x_0)\|} \|u - v\|, \quad (19)$$

for all $u, v \in B(0, r_0)$. Since $\|x - x_0\| < t_*$ thus $\psi'(\|x - x_0\|) < 0$. Hence, (10) together with **h1** imply that

$$\lambda\|f'(x) - f'(x_0)\| \leq \psi'(\|x - x_0\|) - \psi'(0) < 1, \quad \forall x \in B(x_0, t_*).$$

Using assumption in (12), i.e., $t_* \leq r_{x_0}$, last inequality, (19) and **h1**, we concluded that the inequality of the proposition holds, for all $x \in B(x_0, t_*)$. \square

Newton's iteration at a point of a neighborhood of x_0 happens to be a zero of the partial linearization of $f + F$ at such a point. Therefore, we first study the linearization error of f at points in Ω

$$E_f(x, y) := f(y) - [f(x) + f'(x)(y - x)], \quad \forall y, x \in \Omega. \quad (20)$$

In the next result we will bound this error by the linearization error of the majorant function ψ , namely,

$$e_\psi(t, u) := \psi(u) - [\psi(t) + \psi'(t)(u - t)], \quad \forall t, u \in [0, R]. \quad (21)$$

Lemma 9. Take $x, y \in B(x_0, R)$ and $0 \leq t < v < R$. If $\|x - x_0\| \leq t$ and $\|y - x\| \leq v - t$ then

$$\lambda\|E_f(x, y)\| \leq e_\psi(t, v) \frac{\|y - x\|^2}{(v - t)^2} \leq \frac{1}{2} \psi''(v)(v - t)^2. \quad (22)$$

Proof. Since $x + \tau(y - x) \in B(x_0, R)$, for all $\tau \in [0, 1]$ and f is continuously differentiable in Ω , the linearization error of f in (20) is equivalent to

$$E_f(x, y) = \int_0^1 [f'(x + \tau(y - x)) - f'(x)](y - x) d\tau,$$

which combined with the assumption in (10) and after some simple algebraic manipulations we obtain

$$\lambda\|E_f(x, y)\| \leq \int_0^1 [\psi'(\|x - x_0\| + \tau\|y - x\|) - \psi'(\|x - x_0\|)] \|y - x\| d\tau. \quad (23)$$

Using assumption **h2**, we know that ψ' is convex. Thus, since $\|x - x_0\| \leq t$ we conclude that

$$\psi'(\|x - x_0\| + \tau\|y - x\|) - \psi'(\|x - x_0\|) \leq \psi'(t + \tau\|y - x\|) - \psi'(t), \quad \forall \tau \in [0, 1].$$

Due to $\|y - x\| < v - t$ and $v < R$, first statement in Proposition 1 together with last inequality implies

$$\psi'(\|x - x_0\| + \tau\|y - x\|) - \psi'(\|x - x_0\|) \leq [\psi'(t + \tau\|v - t\|) - \psi'(t)] \frac{\|y - x\|}{v - t}, \quad \forall \tau \in [0, 1].$$

Combining the inequality in (23) with last inequality we conclude that

$$\lambda\|E_f(x, y)\| \leq \int_0^1 [\psi'(t + \tau\|v - t\|) - \psi'(t)] \frac{\|y - x\|^2}{v - t} d\tau,$$

which, after performing the integration yields (22). Now, we are going to prove the last inequality in (22). Definition in (21) implies

$$e_\psi(t, v) = \int_0^1 [\psi'(t + \tau(v - t)) - \psi'(t)](v - t) d\tau.$$

We know that ψ' is convex. Thus, using the first and second statement in Proposition 1, it follows from last equality that

$$e_\psi(t, v) \leq \int_0^1 \frac{\psi'(v) - \psi'(t)}{v - t} \tau(v - t)^2 d\tau \leq \int_0^1 \psi''(v) \tau(v - t)^2 d\tau = \frac{1}{2} \psi''(v)(v - t)^2,$$

which, using first inequality in (22) and considering that $\|y - x\| \leq v - t$, gives the desired inequality. \square

Proposition 8 guarantees, in particular, that for each $x \in B(x_0, t_*)$ the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$ and consequently, the Newton iteration mapping is well-defined. Let us call N_{f+F} , the Newton iteration mapping for $f + F$ in that region, namely, $N_{f+F} : B(x_0, t_*) \rightarrow \mathbb{X}$ is defined by

$$N_{f+F}(x) := L_f(x, 0)^{-1} \cap B(x_1, r_{x_1}). \quad (24)$$

Using (4) we conclude that the definition of the Newton iteration mapping in (24) is equivalent to

$$0 \in f(x) + f'(x)(N_{f+F}(x) - x) + F(N_{f+F}(x)), \quad N_{f+F}(x) \in B(x_1, r_{x_1}), \quad \forall x \in B(x_0, t_*). \quad (25)$$

Therefore, one can apply a *single* Newton iteration on any $x \in B(x_0, t_*)$ to obtain $N_{f+F}(x)$ which may not belong to $B(x_0, t_*)$. Thus, this is enough to guarantee the well-definedness of only one iteration. To ensure that Newtonian iterations may be repeated indefinitely or, in particular, invariant on subsets of $B(x_0, t_*)$, we need some additional results. First, define some subsets of $B(x_0, t_*)$ in which, as we shall prove, Newton iteration mapping (24) are “well behaved”. Define

$$K(t) := \left\{ x \in \Omega : \|x - x_0\| \leq t, \quad \|L_f(x, 0)^{-1} \cap B(x_1, r_{x_1}) - x\| \leq -\frac{\psi(t)}{\psi'(t)} \right\}, \quad t \in [0, t_*], \quad (26)$$

$$K := \bigcup_{t \in [0, t_*]} K(t). \quad (27)$$

Proposition 10. *For each $0 \leq t < t_*$ we have $K(t) \subset B(x_0, t_*)$ and $N_{f+F}(K(t)) \subset K(n_\psi(t))$. As a consequence, $K \subseteq B(x_0, t_*)$ and $N_{f+F}(K) \subset K$.*

Proof. The first inclusion follows trivially from the definition of $K(t)$. Take $x \in K(t)$ and, from definitions (26) and (17), follow that

$$\|x - x_0\| \leq t, \quad \|L_f(x, 0)^{-1} \cap B(x_1, r_{x_1}) - x\| \leq -\frac{\psi(t)}{\psi'(t)}, \quad t < n_\psi(t) < t_*. \quad (28)$$

Definition of Newton iteration mapping in (24) implies that, for all $x \in K(t)$ there holds

$$\|N_{f+F}(x) - x_0\| \leq \|x - x_0\| + \|N_{f+F}(x) - x\| = \|x - x_0\| + \|L_f(x, 0)^{-1} \cap B(x_1, r_{x_1}) - x\|,$$

and consequently, using (17) and (28), the last inequality imply that

$$\|N_{f+F}(x) - x_0\| \leq t - \frac{\psi(t)}{\psi'(t)} = n_\psi(t) < t_*. \quad (29)$$

For simplify the notations, let $x_+ = N_{f+F}(x) \in B(x_1, r_{x_1})$. Thus, using (25) and definition in (3) we have

$$0 \in L_f(x, x_+) = f(x) + f'(x)(x_+ - x) + F(x_+).$$

After some simple manipulations in last inequality and taking into account (20) we obtain that

$$\begin{aligned} 0 &\in -f(x_+) + f(x) + f'(x)(x_+ - x) + f(x_+) + f'(x_+)(x_+ - x_+) + F(x_+) \\ &= -E_f(x, x_+) + f(x_+) + f'(x_+)(x_+ - x_+) + F(x_+). \end{aligned}$$

Using (3), we conclude that the last inclusion is equivalent to $E_f(x, x_+) \in L_f(x_+, x_+)$, which implies that

$$x_+ \in L_f(x_+, E_f(x, x_+))^{-1} \cap B(x_1, r_{x_1}). \quad (30)$$

Since the majorant function ψ has a smallest root $t_* \in (0, R)$, we have from (29) that $x_+ \in B[x_0, t_*]$. Now, we are going to prove that

$$E_f(x, x_+) \in B[0, r_0]. \quad (31)$$

Since $x \in K(t)$, definitions (17) and (24) together with (28) imply that $t < n_\psi(t)$ and $\|x_+ - x\| \leq n_\psi(t) - t$. Thus, applying second inequality in Lemma 9 with $y = x_+$ and $v = n_\psi(t)$ we conclude that

$$\lambda \|E_f(x, x_+)\| \leq \frac{1}{2} \psi''(n_\psi(t)) (n_\psi(t) - t)^2.$$

On the other hand, from **h2** we have ψ'' is increasing and Lemma 6 together **h1** gives $n_\psi(t) - t = -\psi(t)/\psi'(t) \leq -\psi(0)/\psi'(0) = \psi(0)$. Thus, above inequality becomes

$$\lambda \|E_f(x, x_+)\| \leq \frac{1}{2} \psi''(t_*) \psi(0)^2.$$

Therefore, using (12) we obtain the desired inclusion in (31). Hence, since $x_+ \in B[x_0, t_*]$, combining (30) with (31) and first part of Proposition 8, we conclude that $x_+ = L_f(x_+, E_f(x, x_+))^{-1} \cap B(x_1, r_{x_1})$. Thus, using the second part of Proposition 8 we have

$$\|L_f(x_+, 0)^{-1} \cap B(x_1, r_{x_1}) - x_+\| \leq -\frac{\lambda}{\psi'(\|x_+ - x_0\|)} \|E_f(x, x_+)\|.$$

Due to $x_+ = N_{f+F}(x)$ we have from (29) that $\|x_+ - x_0\| \leq n_\psi(t)$. Then, taking into account that ψ' is increasing and negative, it follows from above inequality, Lemma 9, (24) and (28) that

$$\|L_f(x_+, 0)^{-1} \cap B(x_1, r_{x_1}) - x_+\| \leq -\frac{\lambda}{\psi'(n_\psi(t))} \|E_f(x, x_+)\| \leq -\frac{e_\psi(t, n_\psi(t))}{\psi'(n_\psi(t))} \frac{\|x_+ - x\|^2}{(n_\psi(t) - t)^2}.$$

On the other hand, using the definition (17) and (21), after some manipulations we conclude that

$$\psi(n_\psi(t)) = \psi(n_\psi(t)) - [\psi(t) + \psi'(t)(n_\psi(t) - t)] = e_\psi(t, n_\psi(t)),$$

and because $x_+ = N_{f+F}(x)$, (17) and the second inequality in (28) imply $\|x - x_+\| \leq n_\psi(t) - t$, above inequality becomes

$$\|L_f(x_+, 0)^{-1} \cap B(x_1, r_{x_1}) - x_+\| \leq -\frac{\psi(n_\psi(t))}{\psi'(n_\psi(t))}.$$

Therefore, since (29) implies $\|x_+ - x_0\| \leq n_\psi(t)$ we conclude that the second inclusion of the proposition is proved.

The third inclusion $K \subseteq B(x_0, t_*)$ follows trivially from (26) and (27). To prove the last inclusion $N_{f+F}(K) \subset K$, take $x \in K$. Thus, $x \in K(t)$ for some $t \in [0, t_*)$. From the second inclusion of the proposition, we have $N_{f+F}(x) \in K(n_\psi(t))$. Since $n_\psi(t) \in [0, t_*)$ and using the definition of K in (27) we conclude the proof. \square

3.2 Convergence analysis

To prove the convergence result, which is a consequence of the above results, firstly we note that the definition (24) implies that the sequence $\{x_k\}$ defined in (13), can be formally stated by

$$x_{k+1} = N_{f+F}(x_k), \quad k = 0, 1, \dots, \quad (32)$$

or equivalently,

$$0 \in f(x_k) + f'(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \quad x_{k+1} \in B(x_1, r_{x_1}), \quad k = 0, 1, \dots \quad (33)$$

First we will show that the sequence generated by Newton method converges to $x_* \in B[x_0, t_*]$, a solution of the generalized equation (1), and is well behaved with respect to the set defined in (26).

Corollary 11. *The sequence $\{x_k\}$ is well defined, is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ satisfying $0 \in f(x_*) + F(x_*)$. Moreover, $x_k \in K(t_k)$, for $k = 0, 1 \dots$ and*

$$\|x_* - x_k\| \leq t_* - t_k, \quad k = 0, 1 \dots$$

Proof. Since the mapping $x \mapsto L_f(x_0, x)$ is strongly regular at x_1 for 0, it follow from (4) and Corollary 4 that $x_1 = L_f(x_0, 0)^{-1} \cap B(x_1, r_{x_1})$ and the first Newton iterate is well defined. Thus, from **h1**, (11) and definitions (26) and (27) we have

$$\{x_0\} = K(0) \subset K. \quad (34)$$

We know from Proposition 10 that $N_{f+F}(K) \subset K$. Thus, using (34) and (32) we conclude that the sequence $\{x_k\}$ is well defined and rests in K . From the first inclusion on second part of the Proposition 10 we have trivially that $\{x_k\}$ is contained in $B(x_0, t_*)$. To prove the convergence, first we are going to prove by induction that

$$x_k \in K(t_k), \quad k = 0, 1 \dots \quad (35)$$

The above inclusion, for $k = 0$, follows from (34). Assume now that $x_k \in K(t_k)$. Then combining Proposition 10, (32) and (17) we conclude that $x_{k+1} \in K(t_{k+1})$, which completes the induction proof. Now, using (35) and (26) we have

$$\|L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}) - x_k\| \leq -\frac{\psi(t_k)}{\psi'(t_k)}, \quad k = 0, 1 \dots,$$

which, combined with (32) and definitions (24) and (13) becomes

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad k = 0, 1 \dots \quad (36)$$

Taking into account that $\{t_k\}$ converges to t_* , we easily conclude from the above inequality that

$$\sum_{k=k_0}^{\infty} \|x_{k+1} - x_k\| \leq \sum_{k=k_0}^{\infty} t_{k+1} - t_k = t_* - t_{k_0} < +\infty,$$

for any $k_0 \in \mathbb{N}$. Hence, we conclude that $\{x_k\}$ is a Cauchy sequence in $B(x_0, t_*)$ and thus it converges to some $x_* \in B[x_0, t_*]$. Therefore, using again (36) we also conclude that the inequality in the corollary holds.

Now, we are going to show that x_* is a solution to the generalized equation $f(x) + F(x) \ni 0$. From inclusion in (33) we conclude

$$(x_{k+1}, -f(x_k) - f'(x_k)(x_{k+1} - x_k)) \in \text{gph } F, \quad k = 0, 1, \dots$$

Since f is continuous with continuous derivative f' in Ω , $B[x_0, t_*] \subset \Omega$ and F has closed graph, last inclusion implies that

$$(x_*, -f(x_*)) = \lim_{k \rightarrow \infty} ((x_{k+1}, -f(x_k) - f'(x_k)(x_{k+1} - x_k)) \in \text{gph } F,$$

which implies $f(x_*) + F(x_*) \ni 0$ and proof is complete. \square

We have already proved that the sequence $\{x_k\}$ converges to a solution x_* of generalized equation $f(x) + F(x) \ni 0$ and $x_* \in B[x_0, t_*]$. Now, we will prove that $\{x_k\}$ converges Q -linearly and that x^* is the unique solution of $f(x) + F(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$. Furthermore, by assuming that ψ satisfies **h4**, we will also prove that $\{x_k\}$ converges Q -quadratically. For that, we need of the following result:

Lemma 12. *Take $x, y \in B(x_0, R)$ and $0 \leq \psi(0) \leq t < R$. If*

$$t < t^*, \quad \|x - x_0\| \leq t, \quad \|y - x_1\| \leq r_{x_1}, \quad \|y - x\| \leq t_* - t, \quad 0 \in f(y) + F(y), \quad (37)$$

then the following inequality holds

$$\|y - N_{f+F}(x)\| \leq [t_* - n_\psi(t)] \frac{\|y - x\|^2}{(t_* - t)^2}.$$

Proof. Since $0 \in f(y) + F(y)$, using (20) and (3), after some simple manipulations we obtain that

$$0 \in f(y) + F(y) = E_f(x, y) + L_f(x, y),$$

which by (4) implies that $y \in L_f(x, -E_f(x, y))^{-1}$. Now, we are going to prove that

$$E_f(x, y) \in B(0, r_0). \quad (38)$$

Applying Lemma 9 with $v = t_*$, and using that $0 \leq \psi(0) \leq t < t_*$ we have

$$\lambda \|E_f(x, y)\| \leq \frac{1}{2} \psi''(t_*) (t_* - t)^2 \leq \frac{1}{2} \psi''(t_*) (t_* - \psi(0))^2.$$

On the other hand, Lemma 6 give us $t_* - n_\psi(0) \leq t_*/2$, which implies that $t_* - n_\psi(0) \leq n_\psi(0) = \psi(0)$. Therefore, above equation becomes

$$\lambda \|E_f(x, y)\| \leq \frac{1}{2} \psi''(t_*) \psi(0)^2,$$

which under assumption in (12) gives the desired inclusion in (38). Since Proposition 8 implies that for any $x \in B(x_0, t^*)$, the mapping $z \mapsto L_f(x, z)^{-1} \cap B(x_1, r_{x_1})$ is single-valued from $B(0, r_0)$ to $B(x_1, r_{x_1})$. Thus, taking into account third inequality in (37), inclusion in (38) and that $y \in L_f(x, -E_f(x, y))^{-1}$, we conclude that $y = L_f(x, -E_f(x, y))^{-1} \cap B(x_1, r_{x_1})$. Therefore, combining (24) with second part of Proposition 8 we conclude

$$\|y - N_{f+F}(x)\| = \|L_f(x, -E_f(x, y))^{-1} \cap B(x_1, r_{x_1}) - L_f(x, 0)^{-1} \cap B(x_1, r_{x_1})\| \leq -\frac{\lambda}{\psi'(t)} \|E_f(x, y)\|,$$

and since $t < t^*$, $\|x - x_0\| \leq t$ and $\|y - x\| \leq t_* - t$, we can apply Lemma 9 with $v = t_*$ to obtain

$$\|y - N_{f+F}(x)\| \leq -\frac{e_\psi(t, t_*)}{\psi'(t)} \frac{\|y - x\|^2}{(t_* - t)^2}.$$

But, due to $0 \leq t < t_*$ and $\psi'(t) < 0$, using (21), (17) and $\psi(t_*) = 0$ we have

$$-\frac{e_\psi(t, t_*)}{\psi'(t)} = t_* - t + \frac{\psi(t)}{\psi'(t)} - \frac{\psi(t_*)}{\psi'(t)} = t_* - t + \frac{\psi(t)}{\psi'(t)} = t_* - n_\psi(t),$$

which combined with last inequality gives the desired result. \square

Corollary 13. *The sequences $\{x_k\}$ and $\{t_k\}$ satisfy the following inequality*

$$\|x_* - x_{k+1}\| \leq \frac{t_* - t_{k+1}}{(t_* - t_k)^2} \|x_* - x_k\|^2, \quad k = 0, 1, \dots \quad (39)$$

As a consequence, the sequence $\{x_k\}$ converges Q -linearly to the solution x_ as follows*

$$\|x_* - x_{k+1}\| \leq \frac{1}{2} \|x_* - x_k\|, \quad k = 0, 1, \dots \quad (40)$$

*Additionally, if ψ satisfies **h4** then the sequence $\{x_k\}$ converges Q -quadratically to x_* as follows*

$$\|x_* - x_{k+1}\| \leq \frac{\psi''(t_*)}{-2\psi'(t_*)} \|x_* - x_k\|^2, \quad k = 0, 1, \dots \quad (41)$$

Proof. We know, from Corollary 11, that $\{x_k\}$ is well defined, converges to x_* , $\|x_k - x_0\| \leq t_k$ and $\|x_* - x_k\| \leq t_* - t_k$, for $k = 0, 1, \dots$. Since $\{x_k\}$ is well defined, it follows from (13) that $x_k \in B(x_1, r_{x_1})$ for $k = 1, 2, \dots$. Hence $x_* \in B[x_1, r_{x_1}]$, i.e., $\|x_* - x_1\| \leq r_{x_1}$. Hence, since **h1** implies $t_1 = n_\psi(0) = \psi(0)$ and $\{t_k\}$ is strictly increasing, we can apply Lemma 12 with $x = x_k$, $y = x_*$ and $t = t_k$ to obtain

$$\|x_* - N_{f+F}(x_k)\| \leq [t_* - n_\psi(t_k)] \frac{\|x_* - x_k\|^2}{(t_* - t_k)^2}.$$

Thus inequality (39) follows from the above inequality, (32) and (18). By the first part in Lemma 6, (18) and Corollary 11 we have

$$\frac{t_* - t_{k+1}}{t_* - t_k} \leq \frac{1}{2}, \quad \frac{\|x_* - x_k\|}{t_* - t_k} \leq 1.$$

Combining these inequalities with (39) we obtain (40). Now, assume that **h4** holds. Then, by Corollary 7, the second inequality on (16) holds, which combined with (39) imply (41). \square

Corollary 14. *The limit x_* of the sequence $\{x_k\}$ is the unique solution of the generalized equation $f(x) + F(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$.*

Proof. Corollary 11 implies that $\{x_k\}$ is well defined and $\{x_k\}$ is contained in $B(x_0, t_*)$, thus it follows from (13) that $x_k \in B(x_0, t_*) \cap B(x_1, r_{x_1})$ for $k = 1, 2, \dots$. Hence $x_* \in B[x_0, t_*] \cap B[x_1, r_{x_1}]$. Suppose there exist $y_* \in B[x_0, t_*] \cap B[x_1, r_{x_1}]$ such that y_* is solution of $f(x) + F(x) \ni 0$. We will prove by induction that

$$\|y_* - x_k\| \leq t_* - t_k, \quad k = 0, 1, \dots \quad (42)$$

The case $k = 0$ is trivial, because $t_0 = 0$ and $y_* \in B[x_0, t_*]$. We assume that the inequality holds for some k . First note that Corollary 11 implies that $x_k \in K(t_k)$, for $k = 0, 1, \dots$. Thus, from definition of $K(t_k)$ we conclude that $\|x_k - x_0\| \leq t_k$, for $k = 0, 1, \dots$. Since **h1** implies $t_1 = n_\psi(0) = \psi(0)$, $\{t_k\}$ is strictly increasing and $\|x_k - x_0\| \leq t_k$, we may apply Lemma 12 with $x = x_k$, $y = y_*$ and $t = t_k$ to obtain

$$\|y_* - N_{f+F}(x_k)\| \leq [t_* - n_\psi(t_k)] \frac{\|y_* - x_k\|^2}{(t_* - t_k)^2}, \quad k = 1, 2, \dots$$

Using inductive hypothesis, (32) and (18) we obtain, from latter inequality, that (42) holds for $k + 1$. Since x_k converges to x_* and t_k converges to t_* , from (42) we conclude that $y_* = x_*$. Therefore, x_* is the unique solution of $f(x) + F(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$. \square

4 Special cases

In this section, we will present some special cases of Theorem 5. It is worth pointing out that to find a majorizing function for a given nonlinear function is a very difficult problem and this is not our aim in this moment. On the other hand, there exist some classes of well known functions which a majorant function is available, below we will present two examples, namely, the classes of functions satisfying a Lipschitz-like and Smale's conditions, respectively. In this sense, the results obtained in Theorem 5 unify the convergence analysis of Newton's method for the classes of generalized equations involving these functions, for instance, Theorem 2 of [18] due to N. H. Josephy and, a particular instance of Theorem 2 of [8] due to A. L. Dontchev and a version of Smale's theorem on Newton's method for analytical functions, see [3].

4.1 Kantorovich's theorem for Newton's method under Lipschitz condition

In this section, we will present a version of the classical Kantorovich's theorem for Newton's method under Lipschitz-type condition for generalized equations. The classical version for $F \equiv \{0\}$ due to L. V. Kantorovich have appeared, for example, in [19], see also [20] and for a historical perspective, see [22].

Theorem 15. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, $\Omega \subseteq \mathbb{X}$ an open set, $f : \Omega \rightarrow \mathbb{Y}$ be continuous with Fréchet derivative f' continuous and $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping with closed graph. Suppose that the partial linearization mapping $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at x_0 is strongly regular at $x_1 \in \Omega$ for 0 with associated Lipschitz constant $\lambda > 0$, and there exists a constant $K > 0$ such that $B(x_0, 1/K) \subset \Omega$ and*

$$\lambda \|f'(y) - f'(x)\| \leq K \|y - x\|, \quad \forall x, y \in B(x_0, 1/K).$$

Moreover, suppose that there exists $b > 0$ such that $bK \leq 1/2$ and

$$\|x_1 - x_0\| \leq b.$$

Additionally, suppose that for r_0 and r_{x_0} fixed in (9) the following inequalities hold:

$$t_* = \frac{1 - \sqrt{1 - 2bK}}{K} \leq r_{x_0}, \quad \frac{K}{2\lambda} b^2 < r_0.$$

Then, the sequence $\{x_k\}$ generated by Newton's method for solving the generalized equation $0 \in f(x) + F(x)$ with starting point x_0 defined by

$$x_{k+1} := L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}), \quad k = 0, 1, \dots,$$

is well defined, $\{x_k\}$ is contained in $B(x_0, t_*)$ and converges to the point $x_* \in B[x_0, t_*]$ which is the unique solution of $f(x) + F(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$, where r_{x_1} is fixed in (9). Moreover, $\{x_k\}$ converges Q -linearly as follows

$$\|x_* - x_{k+1}\| \leq \frac{1}{2} \|x_* - x_k\|, \quad k = 0, 1, \dots$$

Additionally, if $bK < 1/2$ then the sequence $\{x_k\}$ converges Q -quadratically as follows

$$\|x_* - x_{k+1}\| \leq \frac{K}{2\sqrt{1 - 2bK}} \|x_* - x_k\|^2, \quad k = 0, 1, \dots$$

Proof. Since $\psi : [0, 1/K] \rightarrow \mathbb{R}$, defined by $\psi(t) := (K/2)t^2 - t + b$, is a majorant function for f at point x_0 , the result follows by invoking Theorem 5, applied to this particular context. \square

Remark 5. *The above theorem, up to some minor adjustments, merges to classical version, namely, $F \equiv \{0\}$. Indeed, for $F \equiv \{0\}$, the constants in Corollary 4 are $r_0 = r_{x_1} = +\infty$ and $r_{x_0} = t_*$.*

We are going to study an important instance of the generalized equation (1), namely, the generalized equation associated to $F = N_C$, the normal cone of a nonempty, closed and convex subset $C \subset \mathbb{X}$,

$$f(x) + N_C(x) \ni 0. \quad (43)$$

The next result is a version of classical convergence theorem for Newton's method under Lipschitz-type condition for the generalized equation (43), it has been prove by N. H. Josephy in [18].

Theorem 16. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, C a nonempty, closed and convex subset of \mathbb{X} , $\Omega \subseteq \mathbb{X}$ an open set and $f : \Omega \rightarrow \mathbb{Y}$ be continuous with Fréchet derivative f' continuous such that*

$$\|f'(x) - f'(y)\| \leq L\|x - y\|, \quad x, y \in \Omega,$$

where $L > 0$. Moreover, suppose that $f(x_0) + f'(x_0)(x - x_0) + N_C(x)$ is strongly regular at x_1 for 0 with associated Lipschitz constant $\lambda > 0$, $B(x_0, 1/(\lambda K)) \subset \Omega$, there exists $b > 0$ such that $b\lambda L \leq 1/2$ and

$$\|x_1 - x_0\| \leq b.$$

Additionally, suppose that for r_0 and r_{x_0} fixed in (9) the conditions $t_ \leq r_{x_0}$ and $Lb^2/2 < r_0$ hold, where $t_* = (1 - \sqrt{1 - 2b\lambda L})/\lambda L$. Then, the sequence generated by Newton's method, for solving $0 \in f(x) + N_C(x)$, with starting point x_0 ,*

$$x_{k+1} := L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}), \quad k = 0, 1, \dots,$$

is well defined, $\{x_k\}$ is contained in $B(x_0, t_)$ and converges to the point $x_* \in B[x_0, t_*]$ which is the unique solution of $0 \in f(x) + N_C(x)$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$, where r_{x_1} is fixed in (9). Moreover, $\{x_k\}$ converges Q -linearly as follows*

$$\|x_* - x_{k+1}\| \leq \frac{1}{2}\|x_* - x_k\|, \quad k = 0, 1, \dots$$

Additionally, if $b\lambda L < 1/2$ then the sequence $\{x_k\}$ converges Q -quadratically as follows

$$\|x_* - x_{k+1}\| \leq \frac{\lambda L}{2\sqrt{1 - 2b\lambda L}}\|x_* - x_k\|^2, \quad k = 0, 1, \dots$$

Proof. Since $\psi : [0, 1/K) \rightarrow \mathbb{R}$, defined by $\psi(t) := (\lambda L/2)t^2 - t + b$, is a majorant function for f at point x_0 , the result follows by invoking Theorem 15 with $F = N_C$. \square

Remark 6. *The above result contain, as particular instance, several theorem on Newton's method; see, for example, [6, 19].*

A. L. Dontchev [8] under Aubin continuity of the mapping $L_f(x_0, \cdot)^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$L_{f+N_C}(x_0, z)^{-1} := \{y \in \mathbb{R}^n : z \in f(x_0) + f'(x_0)(y - x_0) + N_C(y)\}, \quad (44)$$

has shown that the Newton's method for solving (43) generates a sequence that converges Q -quadratically to a solution. Now, our purpose is to show that, if $\mathbb{X} = \mathbb{R}^m$, $\mathbb{Y} = \mathbb{R}^n$, $F = N_C$ and $C \subset \mathbb{R}^n$ is a nonempty and polyhedral convex set, then in this particular instance, Theorem 2 of [8] follows from Theorem 15. We begin with the formal definition of Aubin continuity; for more details see [9, 10]. First we need the following definitions: The *distance* from a point $v \in \mathbb{R}^n$ to a set $U \subset \mathbb{R}^n$ is $d(v, U) := \inf\{\|v - u\| : u \in U\}$ and the *excess* from the set U to the set V is $e(V, U) := \sup\{d(v, U) : v \in V\}$.

Definition 3. *A mapping $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be Aubin continuous, at $\bar{y} \in \mathbb{R}^m$ for $\bar{x} \in \mathbb{R}^n$, if $\bar{x} \in H(\bar{y})$ and there exist constants $\alpha \geq 0$, $a > 0$ and $c > 0$ such that*

$$e(H(y_1) \cap B(\bar{x}, a), H(y_2)) \leq \alpha\|y_1 - y_2\|, \quad \forall y_1, y_2 \in B(\bar{y}, c).$$

It has been shown in [9, Theorem 1] that if $C \subset \mathbb{R}^n$ is a polyhedral convex set, then Aubin continuity of $L_{f+N_C}(x_0, \cdot)^{-1}$ is equivalent to strong regularity of $f + N_C$. Next we state, with some adjustments, Theorem 2 of [8].

Theorem 17. *Let $C \subset \mathbb{R}^n$ be a polyhedral convex set, $\Omega \subseteq \mathbb{R}^n$ an open set and $f : \Omega \rightarrow \mathbb{Y}$ be continuous with derivative f' continuous such that*

$$\|f'(x) - f'(y)\| \leq L\|x - y\|, \quad \forall x, y \in \Omega,$$

where $L > 0$. Let $x_0 \in \Omega$ and suppose that $\|x_1 - x_0\| \leq b$, $L_{f+N_C}(x_0, \cdot)^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined in (44) is Aubin continuous at $0 \in \mathbb{R}^m$ for $x_1 \in \mathbb{R}^n$ with modulus $\alpha \geq 0$ and associated constants $a > 0$ and $c > 0$, $B(x_0, 1/(\alpha L)) \subset \Omega$ and $\alpha b L \leq 1/2$. Additionally, suppose that for r_0 and r_{x_0} fixed in (9) the conditions $t_* \leq \min\{a, r_{x_0}\}$ and $Lb^2/2 < \min\{c, r_0\}$ hold, where $t_* = (1 - \sqrt{1 - 2\alpha b L})/\alpha L$. Then, the sequence generated by Newton's method, for solving $0 \in f(x) + N_C(x)$, with starting point x_0

$$x_{k+1} := L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}), \quad k = 0, 1, \dots,$$

is well defined, $\{x_k\}$ is contained in $B(x_0, t_*)$ and converges to the point x_* which is the unique solution of $f(x) + N_C(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$, where r_{x_1} is fixed in (9). Moreover, $\{x_k\}$ converges Q -linearly as follows

$$\|x_* - x_{k+1}\| \leq \frac{1}{2}\|x_* - x_k\|, \quad k = 0, 1, \dots$$

Additionally, if $\alpha b L < 1/2$ then the sequence $\{x_k\}$ converges Q -quadratically as follows

$$\|x_* - x_{k+1}\| \leq \frac{\alpha L}{2\sqrt{1 - 2\alpha b L}}\|x_* - x_k\|^2, \quad k = 0, 1, \dots$$

Proof. Since $C \subset \mathbb{R}^n$ is a polyhedral convex set, [9, Theorem 1] implies that Aubin continuity of $L_{f+N_C}(x_0, \cdot)^{-1}$ at $0 \in \mathbb{R}^m$ for $x_1 \in \mathbb{R}^n$ with modulus $\alpha \geq 0$, is equivalent to strongly regularity of $f(x_0) + f'(x_0)(x - x_0) + N_C(x)$ at 0 for x_1 with associated Lipschitz constant $\alpha \geq 0$. Thus, the result follows by applying Theorem 16. \square

4.2 Smale-type theorem for Newton's method

In this section, we will present a version of classical convergence theorem for Newton's method under Smale-type condition for generalized equations, for example, see [3].

Theorem 18. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, $\Omega \subseteq \mathbb{X}$ an open set and $f : \Omega \rightarrow \mathbb{Y}$ be an analytic mapping, $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping with closed graph and $x_0 \in \Omega$. Suppose that the partial linearization mapping $L_f(x_0, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at x_0 , is strongly regular at $x_1 \in \Omega$ for 0 with associated Lipschitz constant $\lambda > 0$ and*

$$\gamma := \sup_{n \geq 1} \left\| \frac{\lambda f^{(n)}(\bar{x})}{n!} \right\|^{1/(n-1)} < +\infty. \quad (45)$$

Moreover, suppose that $B(x_0, 1/\gamma) \subseteq \Omega$ and there exists $b > 0$ such that $\|x_1 - x_0\| \leq b$ and $b\gamma \leq 3 - 2\sqrt{2}$. Additionally, suppose that for r_0 and r_{x_0} fixed in (9) the conditions

$$t_* \leq r_{x_0}, \quad \frac{4^3 \gamma b^2}{\lambda \left(3 - b\gamma + \sqrt{(b\gamma + 1)^2 - 8b\gamma} \right)^3} < r_0, \quad (46)$$

hold, where $t_* = (b\gamma + 1 - \sqrt{(b\gamma + 1)^2 - 8b\gamma})/4\gamma$. Then, the sequence generated by Newton's method for solving $f(x) + F(x) \ni 0$ with starting point x_0 ,

$$x_{k+1} := L_f(x_k, 0)^{-1} \cap B(x_1, r_{x_1}), \quad k = 0, 1, \dots,$$

is well defined, $\{x_k\}$ is contained in $B(x_0, t_*)$ and converges to the point x_* , which is the unique solution of $f(x) + F(x) \ni 0$ in $B[x_0, t_*] \cap B[x_1, r_{x_1}]$, where r_{x_1} is fixed in (9). Moreover, $\{x_k\}$ converges Q -linearly as follows

$$\|x_* - x_{k+1}\| \leq \frac{1}{2} \|x_* - x_k\|, \quad k = 0, 1, \dots$$

Additionally, if $b\gamma < 3 - 2\sqrt{2}$, then $\{x_k\}$ converges Q -quadratically as follows

$$\|x_* - x_{k+1}\| \leq \frac{\gamma}{(1 - \gamma t_*)[2(1 - \gamma t_*)^2 - 1]} \|x_* - x_k\|^2, \quad k = 0, 1, \dots$$

Before proving above theorem we need of two results. The next results gives a condition that is easier to check than condition (10), whenever the mapping under consideration are twice continuously differentiable, and its proof follows the same path of Lemma 21 of [14].

Lemma 19. *Let $\Omega \subset \mathbb{X}$ be an open set, and let $f : \Omega \rightarrow \mathbb{Y}$ be an analytic function. Suppose that $x_0 \in \Omega$ and $B(x_0, 1/\gamma) \subset \Omega$, where γ is defined in (45). Then for all $x \in B(x_0, 1/\gamma)$, it holds that $\|f''(x)\| \leq 2\gamma/(1 - \gamma\|x - x_0\|)^3$.*

The next result gives a relationship between the second derivatives f'' and ψ'' , which allow us to show that f and ψ satisfy (10), and its proof is similar to Lemma 22 of [14].

Lemma 20. *Let \mathbb{X}, \mathbb{Y} be Banach spaces, $\Omega \subseteq \mathbb{X}$ be an open set, $f : \Omega \rightarrow \mathbb{Y}$ be twice continuously differentiable. Let $x_0 \in \Omega$, $R > 0$ and $\kappa = \sup\{t \in [0, R) : B(x_0, t) \subset \Omega\}$. Let $\lambda > 0$ and $\psi : [0, R) \rightarrow \mathbb{R}$ be twice continuously differentiable such that $\lambda\|f''(x)\| \leq \psi''(\|x - x_0\|)$, for all $x \in B(x_0, \kappa)$, then f and ψ satisfy (10).*

[Proof of Theorem 18]. Consider $\psi : [0, 1/\gamma) \rightarrow \mathbb{R}$ defined by $\psi(t) = t/(1 - \gamma t) - 2t + b$. Note that ψ is analytic and $\psi(0) = b > 0$, $\psi'(t) = 1/(1 - \gamma t)^2 - 2$, $\psi'(0) = -1$, $\psi''(t) = 2\gamma/(1 - \gamma t)^3$ and $\psi(t_*) = 0$. It follows from the last equalities that ψ satisfies **h1**, **h2**, **h3** and (46). Combining Lemma 20 with Lemma 19, we conclude that f and ψ satisfy (10). Therefore, the result follows by applying the Theorem 5. \square

5 Final remarks

In this paper we have obtained a semi local convergence result to Newton's method for solving generalized equation in Banach spaces and under the majorant condition. As future works, we propose to study this method using the approach of this paper under a weak assumption than strong regularity, namely, the regularity metric; see [10]. It is well known that the inexact analysis support the efficient computational implementations of the exact ones and, as we have seen above, the majorant condition allowed us to unify several convergence results pertaining to Newton's method. So, unifying result for inexact versions of Newton's method would be very welcome.

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